Extended Abstracts

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Regularity of Lipschitz free boundaries in the Alt–Caffarelli problem

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Resum *(CAT)*
En aquest treball estudiem la regularitat de les fronteres lliures Lipschitz en el problema d'Alt–Caffarelli. Demostrem que les fronteres lliures Lipschitz són $C^{1,\alpha}$ mitjançant la invariància per reescalament del problema i la regularitat inicial Lipschitz de la frontera. A més a més, també provem que les fronteres $C^{1,\alpha}$ són C^∞, cosa que, juntament amb el resultat anterior, implica que les fronteres lliures Lipschitz són C^∞.

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Abstract

Free boundary problems are a subclass of PDEs in which not only do we have to solve a particular PDE, but we also have to find an unknown domain Ω where our solution solves the problem. More precisely, we have a fixed (smooth) domain D and we want to find a pair (Ω, u) such that $\Omega \subset D$ is a domain and $u\colon \Omega \to \mathbb{R}$ is a solution in Ω of the PDE in question. The term *free boundary* refers to $\partial\Omega \cap D$, that is, the piece of the boundary of Ω that falls inside D. The word *free* signifies the fact that the free boundary depends on our solution and will change as soon as our solution does so.

Motivated by its relevance in fields such as fluid mechanics, optimal design problems and electrostatics, the Alt–Caffarelli problem (sometimes also called the one-phase problem or the Bernoulli problem) is a classical example of a free boundary problem. Studied for the first time in [1], this problem consists in finding a nonnegative function u defined in $B_1 = B_1(0)$ solving

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap B_1, \\ \partial_\nu u = 1 & \text{on } \partial\{u > 0\} \cap B_1. \end{cases} \tag{1}$$

In the general notation used previously, our fixed domain is $D = B_1$, $\Omega = \{u > 0\}$, and the free boundary is $\partial\{u > 0\} \cap B_1$. Observe that in (1) we are imposing two boundary conditions on the free boundary: $u = 0$ (implicitly) and $\partial_\nu u = 1$. This type of problem will not have a solution in general since it constitutes an overdetermined PDE problem. However, if the problem can be solved (which is the case for the Alt–Caffarelli problem), then we may expect to prove extra properties of the free boundary $\partial\{u > 0\} \cap B_1$ thanks to the overdetermination of the problem.

In this work we study the Lipschitz regularity of the free boundary $\partial\{u > 0\} \cap B_1$. More precisely, we assume the free boundary to be Lipschitz and then show how to improve its regularity by exploiting the overdetermined nature of the problem. The main result we focus on is the following one:

Theorem. *Let u be a (viscosity) solution of the Alt–Caffarelli problem (1). Assume that the free boundary $\partial\{u > 0\} \cap B_1$ is Lipschitz. Then $\partial\{u > 0\} \cap U$ is smooth for any open set $U \Subset B_1$. Moreover, $u \in C^\infty(\overline{\{u > 0\}} \cap U)$ and u solves (1) in the classical sense in U.*

The proof of this theorem is accomplished in two steps: first, by proving that Lipschitz free boundaries are $C^{1,\alpha}$, and second, by showing that $C^{1,\alpha}$ free boundaries are smooth. For the first step, the main idea is to use the rescaling invariance of (1). Notice that if u is a solution, then for any $r > 0$ the function $u_r(x) = \frac{1}{r}u(rx)$ is also a solution in the corresponding rescaled domain $B_{1/r}$. This property is crucial to finish this first step. Geometrically, the Lipschitz regularity of $\partial\{u > 0\} \cap B_1$ implies that the free boundary always remains outside a cone of a fixed opening. Using that our solution satisfies (1), we are able to show that we can improve the opening of this cone in the ball $B_{1/2}$. However, this alone is clearly insufficient to conclude that the free boundary is $C^{1,\alpha}$. What enables us to complete the proof of this first step is precisely the rescaling invariance of the problem, which we use to repeat the opening improvement iteratively in the sequence of balls $B_{2^{-k}}$. Intuitively, this process tells us that the free boundary "flattens" as we zoom in at the origin which implies, after some extra steps, that the free boundary is $C^{1,\alpha}$.

As for the second step, we perform some computations combined with Schauder estimates for the Laplacian to show that once we have $C^{1,\alpha}$ regularity on the free boundary, then we can improve the boundary as much as we want to obtain C^∞ regularity. Lastly, combining both steps and using a simple covering argument we obtain the proof of the theorem.

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References

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